

L^2 -TORSION INVARIANTS AND THE MAGNUS REPRESENTATION OF THE MAPPING CLASS GROUP

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Dedicated to Professor Shigeyuki Morita on his 60th birthday

ABSTRACT. In this paper, we study a series of L^2 -torsion invariants from the viewpoint of the mapping class group of a surface. We establish some vanishing theorems for them. Moreover we explicitly calculate the first two invariants and compare them with hyperbolic volumes.

1. MAGNUS REPRESENTATION

Let $\Sigma_{g,1}$ be a compact oriented smooth surface of genus g with a boundary $\partial\Sigma_{g,1} \cong S^1$. In this paper, we always assume that $g \geq 1$. We take and fix a base point $*$ in $\partial\Sigma_{g,1}$ of $\Sigma_{g,1}$. Let $\mathcal{M}_{g,1}$ be the mapping class group of $\Sigma_{g,1}$, namely, the group of all isotopy classes of orientation preserving diffeomorphisms of $\Sigma_{g,1}$ relative to the boundary. We denote $\pi_1(\Sigma_{g,1}, *)$ by Γ , which is a free group of rank $2g$, and fix a generating system $\Gamma = \langle x_1, \dots, x_{2g} \rangle$. Let $\mathbb{Z}\Gamma$ be the group ring of Γ over \mathbb{Z} . We write $\varphi_* \in \text{Aut}(\Gamma)$ to the automorphism induced from $\varphi \in \mathcal{M}_{g,1}$. The following result, usually called the Dehn-Nielsen-Baer theorem, is classical and fundamental to study the mapping class group $\mathcal{M}_{g,1}$ by using combinatorial group theories (see [9] Section 2.9).

Proposition 1.1 (Zieschang [27]). *The above induced homomorphism $\mathcal{M}_{g,1} \ni \varphi \mapsto \varphi_* \in \text{Aut}(\Gamma)$ is injective.*

As a corollary, we see that φ can be determined by the words $\varphi_*(x_1), \dots, \varphi_*(x_{2g}) \in \Gamma$. Since the fundamental formula $\gamma = 1 + \sum_{i=1}^{2g} (\partial\gamma/\partial x_i)(x_i - 1)$ holds in $\mathbb{Z}\Gamma$ for any $\gamma \in \Gamma$, the word $\varphi_*(x_j)$ is determined by $\{\partial\varphi_*(x_j)/\partial x_i\}$. Here $\partial/\partial x_i : \mathbb{Z}\Gamma \rightarrow \mathbb{Z}\Gamma$ denotes Fox's free differential. See [1] Section 3.1 for a systematic treatment of the subject. The Magnus representation of the mapping class group is defined as follows.

Definition 1.2. The Magnus representation of $\mathcal{M}_{g,1}$ is defined by the assignment

$$r : \mathcal{M}_{g,1} \ni \varphi \mapsto \left(\overline{\frac{\partial\varphi_*(x_j)}{\partial x_i}} \right)_{ij} \in GL(2g, \mathbb{Z}\Gamma),$$

where $\overline{\sum_g \lambda_g g} = \sum_g \lambda_g g^{-1}$ for any element $\sum_g \lambda_g g \in \mathbb{Z}\Gamma$.

Remark 1.3. By the expression $\gamma = 1 + \sum_i (\partial\gamma/\partial x_i)(x_i - 1)$, it follows that r is injective. However, it is not a group homomorphism, just a crossed homomorphism. According to the practice, we call it simply the Magnus representation of $\mathcal{M}_{g,1}$.

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Now for a matrix $B \in M(n, \mathbb{C})$, let us recall that its characteristic polynomial

$$\det(tI - B)$$

is one of the fundamental tools in the linear algebra. Here I denotes the identity matrix of degree n . If we can define a characteristic polynomial of $r(\varphi)$, it may be useful tool to study the mapping class group. In order to define it for a Magnus matrix $r(\varphi)$, we need to clarify the following two points.

- (1) What is the determinant over a non-commutative group ring?
- (2) What is the meaning of a variable “ t ” in the group?

As an answer to these problems, we can formulate that

- the variable t lives in the fundamental group of the mapping torus of φ ,
- a characteristic polynomial “ \det ”($tI - r(\varphi)$) with respect to the Fuglede-Kadison determinant.

In the later sections, we explain that the characteristic polynomial of $r(\varphi)$ is defined as a real number and it essentially gives the L^2 -torsion and the hyperbolic volume of the mapping torus of φ . Moreover taking the lower central series of the surface group Γ , we obtain a family of Magnus representations, so that we can introduce a sequence of L^2 -torsion invariants as an approximate sequence of the hyperbolic volume.

This paper is organized as follows. In the next section, we briefly recall the definition of the Fuglede-Kadison determinant. In Section 3, we summarize some properties of the L^2 -torsion of 3-manifolds and explain a relation to the Magnus representation. We introduce a sequence of L^2 -torsion invariants for a surface bundle over the circle in Section 4 and give some formulas for them in Section 5. In the last section, we discuss some vanishing theorems for L^2 -torsion invariants.

2. FUGLEDE-KADISON DETERMINANT

In this section, we review the combinatorial definition of the Fuglede-Kadison determinant over a non-commutative group ring and its basic properties (see [19] for details).

The idea to define a determinant over a group ring comes from the following observation. That is, for a matrix $B \in GL(n, \mathbb{C})$ with the (non-zero) eigenvalues $\lambda_1, \dots, \lambda_n$, we can formally calculate

$$\begin{aligned} \log |\det B|^2 &= \log \prod_{i=1}^n \lambda_i \bar{\lambda}_i = \sum_{i=1}^n \log \lambda_i \bar{\lambda}_i \\ &= \sum_{i=1}^n \left(\sum_{p=1}^{\infty} \frac{(-1)^{p+1}}{p} (\lambda_i \bar{\lambda}_i - 1)^p \right) \\ &= - \sum_{p=1}^{\infty} \left(\sum_{i=1}^n \frac{1}{p} (1 - \lambda_i \bar{\lambda}_i)^p \right) \\ &= - \sum_{p=1}^{\infty} \frac{1}{p} \operatorname{tr} ((I - BB^*)^p) \end{aligned}$$

by the power series expansion of \log , where B^* is the adjoint matrix of B . More precisely, if we take a sufficiently large constant $K > 0$, we obtain

$$|\det B| = K^n \exp \left(-\frac{1}{2} \sum_{p=1}^{\infty} \frac{1}{p} \operatorname{tr} \left((I - K^{-2} B B^*)^p \right) \right) \in \mathbb{R}_{>0}.$$

Thus if we can define a certain “trace” over a group ring, we get a determinant by using this formula.

Let π be a discrete group and $\mathbb{C}\pi$ denote its group ring over \mathbb{C} . For an element $\sum_{g \in \pi} \lambda_g g \in \mathbb{C}\pi$, we define the $\mathbb{C}\pi$ -trace $\operatorname{tr}_{\mathbb{C}\pi} : \mathbb{C}\pi \rightarrow \mathbb{C}$ by

$$\operatorname{tr}_{\mathbb{C}\pi} \left(\sum_{g \in \pi} \lambda_g g \right) = \lambda_e \in \mathbb{C},$$

where e is the unit element in π . For an $n \times n$ -matrix $B = (b_{ij}) \in M(n, \mathbb{C}\pi)$, we extend the definition of $\mathbb{C}\pi$ -trace by means of

$$\operatorname{tr}_{\mathbb{C}\pi}(B) = \sum_{i=1}^n \operatorname{tr}_{\mathbb{C}\pi}(b_{ii}).$$

Next let us recall the definition of the L^2 -Betti number of an $n \times m$ -matrix $B \in M(n, m, \mathbb{C}\pi)$. We consider the bounded π -equivariant operator

$$R_B : \oplus_{i=1}^n l^2(\pi) \rightarrow \oplus_{i=1}^m l^2(\pi)$$

defined by the natural right action of B . Here $l^2(\pi)$ is the complex Hilbert space of the formal sums $\sum_{g \in \pi} \lambda_g g$ which are square summable. We fix a positive real number K so that $K \geq \|R_B\|_{\infty}$ holds, where $\|R_B\|_{\infty}$ is the operator norm of R_B .

Definition 2.1. The L^2 -Betti number of a matrix $B \in M(n, m, \mathbb{C}\pi)$ is defined by

$$b(B) = \lim_{p \rightarrow \infty} \operatorname{tr}_{\mathbb{C}\pi} \left((I - K^{-2} B B^*)^p \right) \in \mathbb{R}_{\geq 0},$$

where $B^* = (\overline{b_{ji}})$ and $\overline{\sum \lambda_g g} = \sum \overline{\lambda_g} g^{-1}$ for each entry.

Roughly speaking, the L^2 -Betti number $b(B)$ measures the size of the kernel of a matrix B . Hereafter we assume $b(B) = 0$. Then, for a matrix with coefficients in a non-commutative group ring, we can introduce the desired determinant as follows.

Definition 2.2. The Fuglede-Kadison determinant of a matrix $B \in M(n, m, \mathbb{C}\pi)$ is defined by

$$\det_{\mathbb{C}\pi}(B) = K^n \exp \left(-\frac{1}{2} \sum_{p=1}^{\infty} \frac{1}{p} \operatorname{tr}_{\mathbb{C}\pi} \left((I - K^{-2} B B^*)^p \right) \right) \in \mathbb{R}_{>0},$$

if the infinite sum of non-negative real numbers in the above exponential converges to a real number.

Remark 2.3. It is shown that the L^2 -Betti number $b(B)$ and the Fuglede-Kadison determinant $\det_{\mathbb{C}\pi}(B)$ are independent of the choice of the constant K (see [16] for example).

Here we consider the condition of the convergence. For any matrix $B \in M(n, \mathbb{C})$, the condition

$$\lim_{p \rightarrow \infty} \operatorname{tr} \left((I - K^{-2} B B^*)^p \right) = 0$$

implies that B has no zero eigenvalues, and then $|\det B|$ converges. In the case of group rings, if $\det_{\mathbb{C}\pi}(B)$ converges, then $b(B) = 0$. But it is not a sufficient condition, so that we need additional one. It is a problem to decide when $\det_{\mathbb{C}\pi}(B)$ converges. Under the assumption that $b(B) = 0$, such a sufficient condition is given by the positivity of the Novikov-Shubin invariant $\alpha(B)$. Then the convergence of the infinite sum in the Fuglede-Kadison determinant is guaranteed. The Novikov-Shubin invariant of an operator R_B measures how concentrated the spectrum of $R_B^* R_B$ is. However, in general, it is hard to check the positivity of the Novikov-Shubin invariant.

To avoid the difficulty, we need to consider the determinant class condition for groups (see [19], [24] for details). A group π is of $\det \geq 1$ -class if for any $B \in M(n, m, \mathbb{Z}\pi)$ the Fuglede-Kadison determinant of B satisfies $\det_{\mathbb{C}\pi}(B) \geq 1$. There are no known examples of groups which are not of $\det \geq 1$ -class. Further recently it was proved that there is a certain large class \mathcal{G} of groups for which they are of $\det \geq 1$ -class. It includes amenable groups and countable residually finite groups. If we can see that π belongs to \mathcal{G} , namely it is of $\det \geq 1$ -class, the convergence of the Fuglede-Kadison determinant is guaranteed when the L^2 -Betti number is vanishing. See [18], [19], [24] for definitions and properties of these subjects.

3. L^2 -TORSION OF 3-MANIFOLDS

In this section, we quickly recall the definition of the L^2 -torsion of 3-manifolds. It is an L^2 -analogue of the Reidemeister and the Ray-Singer torsion and essentially gives Gromov's simplicial volume under certain general conditions [2], [3], [4], [8], [14], [15], [20], [21], [22]. See [19] and its references for historical background, related works and so on.

Let M be a compact connected orientable 3-manifold. We fix a CW -complex structure on M . We may assume that the action of $\pi_1 M$ on the universal covering \widetilde{M} is cellular (if necessary, we have only to take a subdivision of the original structure). We consider the $\mathbb{C}\pi_1 M$ -chain complex

$$0 \longrightarrow C_3(\widetilde{M}, \mathbb{C}) \xrightarrow{\partial_3} C_2(\widetilde{M}, \mathbb{C}) \xrightarrow{\partial_2} C_1(\widetilde{M}, \mathbb{C}) \xrightarrow{\partial_1} C_0(\widetilde{M}, \mathbb{C}) \longrightarrow 0$$

of \widetilde{M} . Since the boundary operator ∂_i is a matrix with coefficients in $\mathbb{C}\pi_1 M$, if we take the adjoint operator $\partial_i^* : C_{i-1}(\widetilde{M}, \mathbb{C}) \rightarrow C_i(\widetilde{M}, \mathbb{C})$ as in the previous section, we can define the i th (combinatorial) Laplace operator $\Delta_i : C_i(\widetilde{M}, \mathbb{C}) \rightarrow C_i(\widetilde{M}, \mathbb{C})$ by

$$\Delta_i = \partial_{i+1} \circ \partial_{i+1}^* + \partial_i^* \circ \partial_i.$$

Let us suppose that all the L^2 -Betti numbers $b(\Delta_i)$ vanish and the fundamental group $\pi_1 M$ is of $\det \geq 1$ -class. Thereby as a generalization of the classical Reidemeister torsion, the L^2 -torsion $\tau(M)$ is defined by

Definition 3.1.

$$\tau(M) = \prod_{i=0}^3 \det_{\mathbb{C}\pi_1 M}(\Delta_i)^{(-1)^{i+1}i} \in \mathbb{R}_{>0}.$$

As for the positivity of Novikov-Shubin invariants $\alpha(\Delta_i)$ for the Laplace operator Δ_i , it is known that $\alpha(\Delta_i) > 0$ holds under some general assumptions (see [15]). For example, if a compact connected orientable 3-manifold M satisfies

- (1) $\pi_1 M$ is infinite,

- (2) M is an irreducible 3-manifold or $S^1 \times S^2$ or $\mathbb{R}P^3 \# \mathbb{R}P^3$,
- (3) if $\partial M \neq \emptyset$, it consists of tori,
- (4) if $\partial M = \emptyset$, M is finitely covered by a 3-manifold which is a hyperbolic, Seifert or Haken 3-manifold,

then $b(\Delta_i) = 0$ and $\alpha(\Delta_i) > 0$ for each i . Therefore, we see that the L^2 -torsion $\tau(M)$ is also well-defined in view of these conditions.

Remark 3.2. The above condition (4) is automatically satisfied by Perelman's proof of Thurston's Geometrization Conjecture.

As a notable property of the L^2 -torsion, it is known that $\log \tau(M)$ can be interpreted as Gromov's simplicial volume $||M||$ and hyperbolic volume $\text{vol}(M)$ (see [7]) of M . See [21] for the heart of the proof.

Theorem 3.3. *Let M be a compact connected orientable irreducible 3-manifold with an infinite fundamental group such that ∂M is empty or a disjoint union of incompressible tori. Then it holds that*

$$\log \tau(M) = C ||M||,$$

where C is the universal constant not depending on M . In particular, if M is a hyperbolic 3-manifold, we obtain

$$\log \tau(M) = -\frac{1}{3\pi} \text{vol}(M).$$

Next we review Lück's formula for the L^2 -torsion of 3-manifolds ([16] Theorem 2.4). From this formula, we see that $\log \tau$ is a characteristic polynomial of the Magnus representation of the mapping class group.

Theorem 3.4. *Let M be as in the above theorem. We suppose that ∂M is non-empty and $\pi_1 M$ has a deficiency one presentation*

$$\langle s_1, \dots, s_{n+1} \mid r_1, \dots, r_n \rangle.$$

Put A to be the $n \times n$ -matrix with entries in $\mathbb{Z}\pi_1 M$ obtained from the matrix $(\partial r_i / \partial s_j)$ by deleting one of the columns. Then the logarithm of the L^2 -torsion of M is given by

$$\begin{aligned} \log \tau(M) &= -2 \log \det_{\mathbb{C}\pi_1 M}(A) \\ &= -2n \log K + \sum_{p=1}^{\infty} \frac{1}{p} \text{tr}_{\mathbb{C}\pi_1 M} \left((I - K^{-2} A A^*)^p \right), \end{aligned}$$

where K is a constant satisfying $K \geq ||R_A||_{\infty}$.

To see a relation between the Magnus representation and the L^2 -torsion, we describe the above Lück's formula for a surface bundle over the circle.

For an orientation preserving diffeomorphism φ of $\Sigma_{g,1}$, we form the mapping torus M_{φ} by taking the product $\Sigma_{g,1} \times [0, 1]$ and gluing $\Sigma_{g,1} \times \{0\}$ and $\Sigma_{g,1} \times \{1\}$ via φ . This gives a surface bundle over S^1 . Its diffeomorphism type is determined by the monodromy map φ , and conversely the monodromy map φ is determined by a given surface bundle up to conjugacy and isotopy. Here an isotopy fixes setwisely the points on the boundary $\partial \Sigma_{g,1}$. We take a deficiency one presentation of the fundamental group $\pi = \pi_1(M_{\varphi}, *)$,

$$\pi = \langle x_1, \dots, x_{2g}, t \mid r_i : t x_i t^{-1} = \varphi_*(x_i), 1 \leq i \leq 2g \rangle,$$

where the base point $*$ of π and $\Gamma = \pi_1(\Sigma_{g,1}, *)$ is the same one on the fiber $\Sigma_{g,1} \times \{0\} \subset M_\varphi$ and $\varphi_* : \Gamma \rightarrow \Gamma$ is the automorphism induced by $\varphi : \Sigma_{g,1} \rightarrow \Sigma_{g,1}$. It should be noted that π is isomorphic to the semi-direct product of Γ and $\pi_1 S^1 \cong \mathbb{Z} = \langle t \rangle$.

Applying the free differential calculus to the relations r_i ($1 \leq i \leq 2g$), we obtain the Alexander matrix

$$A = \left(\frac{\partial r_i}{\partial x_j} \right) \in M(2g, \mathbb{Z}\pi).$$

Then Lück's formula for a surface bundle over the circle is given by

$$\begin{aligned} \log \tau(M_\varphi) &= -2 \log \det_{\mathbb{C}\pi}(A) \\ &= -4g \log K + \sum_{p=1}^{\infty} \frac{1}{p} \operatorname{tr}_{\mathbb{C}\pi} \left((I - K^{-2} A A^*)^p \right), \end{aligned}$$

where K is a constant satisfying $K \geq \|R_A\|_\infty$.

This formula enables us to interpret the L^2 -torsion $\log \tau$ of a surface bundle over the circle as the characteristic polynomial of the Magnus representation $r(\varphi)$. In fact, an easy calculation shows that

$$A = \left(\frac{\partial r_i}{\partial x_j} \right) = tI - {}^t \overline{r(\varphi)}.$$

Then if we take the Fuglede-Kadison determinant in $M(2g, \mathbb{C}\pi)$, we have

$$\begin{aligned} \det_{\mathbb{C}\pi} \left(tI - {}^t \overline{r(\varphi)} \right) &= \det_{\mathbb{C}\pi} \left(tI - {}^t \overline{r(\varphi)} \right)^* \\ &= \det_{\mathbb{C}\pi} (t^{-1}I - r(\varphi)) \end{aligned}$$

because $\operatorname{tr}_{\mathbb{C}\pi}(BB^*) = \operatorname{tr}_{\mathbb{C}\pi}(B^*B)$ holds. Therefore the L^2 -torsion is interpreted as the characteristic polynomial of $r(\varphi)$.

4. DEFINITION OF L^2 -TORSION INVARIANTS

As was seen in Section 3, Lück's formula gives a way to calculate the simplicial volume from a presentation of the fundamental group. However, in general, it seems to be difficult to evaluate the exact values from the formula. In this section, we introduce a sequence of L^2 -torsion invariants which approximates the original one for a surface bundle over the circle. See [12] for details.

In order to construct such a sequence of L^2 -torsion invariants, we consider the lower central series of Γ . Namely, we take the descending infinite sequence

$$\Gamma_1 = \Gamma \supset \Gamma_2 \supset \cdots \supset \Gamma_k \supset \cdots,$$

where $\Gamma_k = [\Gamma_{k-1}, \Gamma_1]$ for $k \geq 2$. Let N_k be the k th nilpotent quotient $N_k = \Gamma/\Gamma_k$ and $p_k : \Gamma \rightarrow N_k$ be the natural projection.

In the previous section, we considered a chain complex $C_*(\widetilde{M}_\varphi, \mathbb{C})$ of $\mathbb{C}\pi$ -modules. Instead of this complex, we can use the chain complex

$$C_*(M_\varphi, l^2(\pi)) = l^2(\pi) \otimes_{\mathbb{C}\pi} C_*(\widetilde{M}_\varphi, \mathbb{C})$$

to define the same L^2 -torsion $\tau(M_\varphi)$. This point of view allows us to introduce a sequence of the L^2 -torsion invariants.

The group Γ_k is a normal subgroup of π , so that we can take the quotient group $\pi(k) = \pi/\Gamma_k$. It should be noted that $\pi(k)$ is isomorphic to the semi-direct product

$N_k \rtimes \mathbb{Z}$. We denote the induced projection $\pi \rightarrow \pi(k)$ by the same letter p_k . Thereby we can consider the chain complex

$$C_*(M_\varphi, l^2(\pi(k))) = l^2(\pi(k)) \otimes_{\mathbb{C}\pi} C_*(\widetilde{M}_\varphi, \mathbb{C})$$

through the projection p_k . By using the Laplace operator

$$\Delta_i^{(k)} : C_i(M_\varphi, l^2(\pi(k))) \rightarrow C_i(M_\varphi, l^2(\pi(k)))$$

on this complex, we can formally define the k th L^2 -torsion invariant $\tau_k(M_\varphi)$ as follows.

Definition 4.1.

$$\tau_k(M_\varphi) = \prod_{i=0}^3 \det_{\mathbb{C}\pi(k)}(\Delta_i^{(k)})^{(-1)^{i+1}i}.$$

Of course, this definition is well-defined if every L^2 -Betti number $b(\Delta_i^{(k)})$ vanishes and every $\pi(k)$ is of $\det \geq 1$ -class. The next lemma is easily proved (see [12], [17]).

Lemma 4.2. *The L^2 -Betti numbers of $\Delta_i^{(k)}$ are all zero.*

Recall the class \mathcal{G} of groups. It is the smallest class of groups which contains the trivial group and is closed under the following processes: (i) amenable quotients, (ii) colimits, (iii) inverse limits, (iv) subgroups and (v) quotients with finite kernel (see [19], [24]). It is known that \mathcal{G} contains all amenable groups. By definition, $N_k = \Gamma/\Gamma_k$ is a nilpotent group and in particular an amenable group. Hence every N_k belongs to \mathcal{G} . Further for any automorphism $\varphi_* : N_k \rightarrow N_k$, its mapping torus extension (HNN -extension) $N_k \rtimes \mathbb{Z}$ also belongs to \mathcal{G} . Therefore we have

Lemma 4.3. *The group $\pi(k)$ belongs to \mathcal{G} .*

As a result, we can conclude that our L^2 -torsion invariants τ_k can be defined for any $k \geq 1$ and they are all homotopy invariants (see [19], [24]).

Now let us describe a formula of the k th L^2 -torsion invariant $\tau_k(M_\varphi)$ and establish a relation to the Magnus representation of the mapping class group. Let $p_{k*} : \mathbb{C}\pi \rightarrow \mathbb{C}\pi(k)$ be an induced homomorphism over the group rings. For $k \geq 1$, we put

$$A_k = \left(p_{k*} \left(\frac{\partial r_i}{\partial x_j} \right) \right) \in M(2g, \mathbb{C}\pi(k)).$$

Moreover we fix a constant K_k satisfying $K_k \geq \|R_{A_k}\|_\infty$. Then we have

$$\begin{aligned} \log \tau_k(M_\varphi) &= -2 \log \det_{\mathbb{C}\pi(k)}(R_{A_k}) \\ &= -4g \log K_k + \sum_{p=1}^{\infty} \frac{1}{p} \operatorname{tr}_{\mathbb{C}\pi(k)} \left((I - K_k^{-2} A_k A_k^*)^p \right), \end{aligned}$$

by virtue of the same argument as Theorem 3.4.

For the k th invariant τ_k , we have taken the lower central series $\{\Gamma_k\}$ of Γ and the nilpotent quotients $\{N_k\}$. These quotients induce a sequence of representations (more precisely, crossed homomorphisms)

$$r_k : \mathcal{M}_{g,1} \rightarrow GL(2g, \mathbb{Z}N_k)$$

for $k \geq 1$ (see [23]). They naively approximate the original Magnus representation $r : \mathcal{M}_{g,1} \rightarrow GL(2g, \mathbb{Z}\Gamma)$. By the similar observation as before, the k th invariant

$\log \tau_k(M_\varphi)$ can be regarded as the characteristic polynomial of $r_k(\varphi)$ with respect to the Fuglede-Kadison determinant in $M(2g, \mathbb{C}\pi(k))$.

From the viewpoint of the Magnus representation of the mapping class group, it seems natural to raise the following problem.

Problem 4.4. *Show that the sequence $\{\tau_k(M_\varphi)\}$ converges to $\tau(M_\varphi)$ when we take the limit on k .*

In general, such an approximation problem for the L^2 -torsion seems to be difficult. However, similar convergence results are known for the L^2 -Betti numbers. In fact, Lück shows in [18] a theorem relating L^2 -Betti numbers to ordinary Betti numbers of finite coverings. This result is generalized to more general settings by Schick in [24].

As for the Fuglede-Kadison determinant, Lück proves in [19] the following. Let $f : \mathbb{Q}[\mathbb{Z}] \rightarrow \mathbb{Q}[\mathbb{Z}]$ be the $\mathbb{Q}[\mathbb{Z}]$ -map given by multiplication with $p(t) \in \mathbb{Q}[\mathbb{Z}]$ and $f_{(2)} : l^2(\mathbb{Z}) \rightarrow l^2(\mathbb{Z})$ be the linear operator obtained from f by tensoring with $l^2(\mathbb{Z})$ over $\mathbb{Q}[\mathbb{Z}]$. Further let $f_{[n]} : \mathbb{C}[\mathbb{Z}/n] \rightarrow \mathbb{C}[\mathbb{Z}/n]$ be the linear operator obtained from f by taking the tensor product with $\mathbb{C}[\mathbb{Z}/n]$ over $\mathbb{Q}[\mathbb{Z}]$. We then get an approximation result:

$$\log \det_{\mathbb{C}[\mathbb{Z}]}(f_{(2)}) = \lim_{n \rightarrow \infty} \frac{\log \det_{\mathbb{C}[\mathbb{Z}/n]}(f_{[n]})}{n}$$

(see [11] for a similar statement). In [19] Lück also points out that there exists a purely algebraic example where Fuglede-Kadison determinants do not converge.

On the other hand, in general, we have at least an inequality for the Fuglede-Kadison determinant in the limit statement (see [24]). That is, for the operator R_{A_k} we see that

$$\log \det_{\mathbb{C}\pi}(R_A) \geq \limsup_k \log \det_{\mathbb{C}\pi(k)}(R_{A_k})$$

holds. In the last section, we shall discuss Problem 4.4 again and give an affirmative answer under certain conditions.

5. FORMULAS OF τ_1 AND τ_2

In this section, we give explicit formulas of the first two invariants of a sequence of our L^2 -torsion invariants. They are really computable formulas, so that we can make a systematic calculation for low genus cases. In particular, we compare them with hyperbolic volumes. The results discussed here are a summary of our previous paper [12] (see also [10], [11]).

First we consider the Magnus representation

$$r_1 : \mathcal{M}_{g,1} \rightarrow GL(2g, \mathbb{Z}N_1).$$

Here $N_1 = \Gamma/\Gamma_1$ is the trivial group and then the above representation is the same as the usual homological action of $\mathcal{M}_{g,1}$ on $H_1(\Sigma_{g,1}, \mathbb{Z})$. Namely we have the representation

$$r_1 : \mathcal{M}_{g,1} \rightarrow \text{Aut}(H_1(\Sigma_{g,1}, \mathbb{Z}), \langle \ , \ \rangle) \cong \text{Sp}(2g, \mathbb{Z}),$$

where $\langle \ , \ \rangle$ denotes the intersection form on the first homology group. Further $\pi(1) = \pi/\Gamma_1 \cong \mathbb{Z} = \langle t \rangle$ and its group ring $\mathbb{C}\langle t \rangle$ is a commutative Laurent polynomial ring $\mathbb{C}[t, t^{-1}]$. Then the matrix A_1 is nothing but the usual characteristic matrix

of ${}^t r_1(\varphi)$. In this case, it is described by the usual determinant for a matrix with commutative entries.

In order to state the theorem, we recall a definition from number theory (see [6] and its references). For a Laurent polynomial $F(\mathbf{t}) \in \mathbb{C}[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$, the Mahler measure of F is defined by

$$m(F) = \int_0^1 \cdots \int_0^1 \log \left| F(e^{2\pi\sqrt{-1}\theta_1}, \dots, e^{2\pi\sqrt{-1}\theta_n}) \right| d\theta_1 \cdots d\theta_n,$$

where we assume that undefined terms are omitted. Namely we define the integrand to be zero whenever we hit a zero of F .

Theorem 5.1 ([12]). *The logarithm of the first invariant τ_1 is given by*

$$\log \tau_1(M_\varphi) = -2m(\Delta_{r_1(\varphi)}),$$

where $\Delta_{r_1(\varphi)}(t) = \det A_1 = \det(tI - r_1(\varphi))$. Moreover if $\Delta_{r_1(\varphi)}(t)$ has a factorization $\Delta_{r_1(\varphi)}(t) = \prod_{i=1}^{2g} (t - \alpha_i)$ ($\alpha_i \in \mathbb{C}$), then we have

$$\log \tau_1(M_\varphi) = -2 \sum_{i=1}^{2g} \log \max\{1, |\alpha_i|\}.$$

Remark 5.2. In other words, $\log \tau_1(M_\varphi)$ is given by the integral of the Alexander polynomial of M_φ over the circle S^1 (see [16], for the exterior of a knot K in the 3-sphere S^3). Further, $\log \tau_1(M_\varphi)$ can be described by the asymptotic behavior of the order of the first homology group of a cyclic covering (see [11]).

The point of the proof is to identify the Hilbert space $l^2(\mathbb{Z})$ with $L^2(\mathbb{R}/\mathbb{Z})$ in terms of the Fourier transforms. Then the $\mathbb{C}\langle t \rangle$ -trace $\text{tr}_{\mathbb{C}\langle t \rangle} : l^2(\mathbb{Z}) \rightarrow \mathbb{C}$ can be realized as the integration

$$L^2(\mathbb{R}/\mathbb{Z}) \ni f(\theta) \mapsto \int_0^1 f(\theta) d\theta \in \mathbb{C}$$

(see [12] for details). From this description and Kronecker's theorem ([6] Theorem 2), we obtain a certain vanishing theorem of the first invariant.

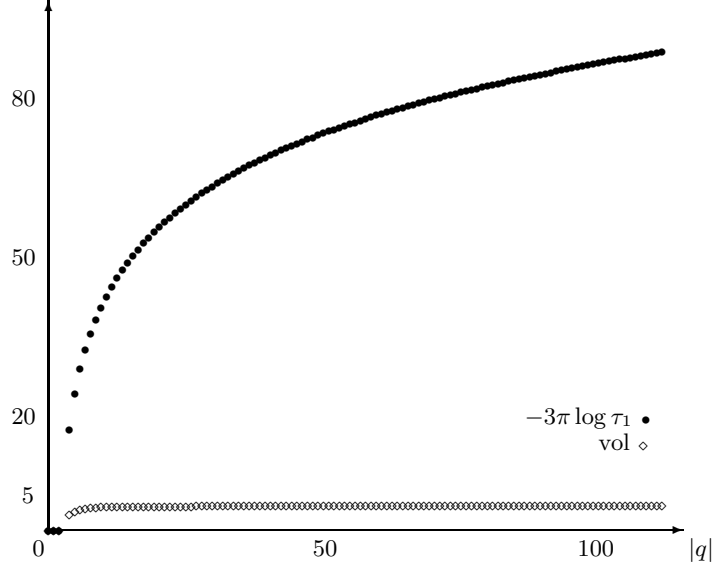
Corollary 5.3. *The logarithm of $\tau_1(M_\varphi)$ vanishes if and only if every eigenvalue of $r_1(\varphi) \in \text{Sp}(2g, \mathbb{Z})$ is a root of unity.*

This corollary seems to be interesting from the viewpoint of Problem 4.4. Because in some case, we can say that the first invariant τ_1 already approximates the simplicial volume. In particular, Corollary 5.3 implies that a torus bundle M_φ ($g = 1$) with a hyperbolic structure (namely, $|\text{tr}(r_1(\varphi))| \geq 3$) has always non-trivial L^2 -torsion invariant $\tau_1(M_\varphi)$. Summing up, we have

Corollary 5.4. *For any $\varphi \in \mathcal{M}_{1,1}$, its mapping torus M_φ admits a hyperbolic structure if and only if M_φ has a non-trivial L^2 -torsion invariant $\tau_1(M_\varphi)$.*

Therefore, the first invariant τ_1 already approximates the simplicial volume in genus one case.

Remark 5.5. It is known that if the characteristic polynomial of $r_1(\varphi) \in \text{Sp}(2g, \mathbb{Z})$ is irreducible over \mathbb{Z} , has no roots of unity as eigenvalues and is not a polynomial in t^n for any $n > 1$, then φ is pseudo-Anosov (see Casson-Bleiler [5]). In this case, $\text{vol}(M_\varphi) \neq 0$ and further $\log \tau_1(M_\varphi) \neq 0$ by Corollary 5.3.

FIGURE 1. $-3\pi \log \tau_1(M_\varphi)$ and $\text{vol}(M_\varphi)$ vs. $|q|$

Example 5.6. It is well-known that the mapping class group of the two dimensional torus $T^2 = \mathbb{R}^2/\mathbb{Z}^2$ is isomorphic to $SL(2, \mathbb{Z})$. Taking a matrix $\begin{pmatrix} q & 1 \\ -1 & 0 \end{pmatrix} \in SL(2, \mathbb{Z})$, it gives a diffeomorphism φ on T^2 . We may assume that it is the identity on some embedded disk by an isotopic deformation and it gives an element of $\mathcal{M}_{1,1}$. We use the same symbol φ for this mapping class. An easy calculation shows that

$$r_1(\varphi) = \begin{pmatrix} q & 1 \\ -1 & 0 \end{pmatrix}$$

and

$$\Delta_{r_1(\varphi)}(t) = \det(tI - r_1(\varphi)) = t^2 - qt + 1.$$

We put $\xi_{\pm} = (q \pm \sqrt{q^2 - 4})/2$ (the eigenvalues of the matrix $r_1(\varphi)$). If $|q| \leq 2$, then $|\xi_{\pm}| = 1$. Hence $\log \tau_1(M_\varphi) = 0$ in these cases. On the other hand, either $|\xi_+|$ or $|\xi_-|$ is greater than one when $|q| \geq 3$, so that M_φ has a non-trivial L^2 -torsion invariant τ_1 in these cases. In fact, the logarithm of the first invariant is given by

$$\log \tau_1(M_\varphi) = -2 \log \max \{|\xi_+|, |\xi_-|\}.$$

The values of $\log \tau_1$ for the traces q and $-q$ are the same, so that it is a function of $|\text{tr}(r_1(\varphi))|$. We put a graph of the L^2 -torsion invariant $-3\pi \log \tau_1(M_\varphi)$ and the hyperbolic volume $\text{vol}(M_\varphi)$ as a function of $|q|$ in Fig 1.

Example 5.7. Next we consider the genus two case. Let t_1, \dots, t_5 be the Lickorish-Humphries generators of $\mathcal{M}_{2,1}$. We take the element $\varphi = t_1 t_3 t_5^2 t_2^{-1} t_4^{-1} \in \mathcal{M}_{2,1}$. As was shown in [5], the characteristic polynomial of $r(\varphi)$ is

$$\begin{aligned} \Delta_{r_1(\varphi)}(t) &= \det(tI - r_1(\varphi)) \\ &= t^4 - 9t^3 + 21t^2 - 9t + 1 \end{aligned}$$

and irreducible over \mathbb{Z} . Moreover it has no roots of unity as zeros. Hence, φ is pseudo-Anosov and M_φ has a non-trivial L^2 -torsion invariant $\tau_1(M_\varphi)$. In fact, we

have

$$-3\pi \log \tau_1(M_\varphi) = 52.954\dots \quad \text{and} \quad \text{vol}(M_\varphi) = 11.466\dots$$

Remark 5.8. In the above two examples, we used SnapPea [26] to compute the hyperbolic volumes.

Now in the following, we consider the second invariant τ_2 . In the case of genus one, we can prove the vanishing of $\log \tau_2(M_\varphi)$.

Theorem 5.9 ([11]). $\log \tau_2(M_\varphi) = 0$ for any $\varphi \in \mathcal{M}_{1,1}$.

This follows from the fact that the group $\pi(2)$ is isomorphic to the fundamental group of a closed torus bundle over the circle. Such a 3-manifold admits no hyperbolic structures, so that the original L^2 -torsion is trivial and we obtain the assertion.

On the other hand, in the case of $g \geq 2$, it is difficult to describe $\log \tau_2$ explicitly on the full mapping class group $\mathcal{M}_{g,1}$. However, we can do it on the Torelli group. Let φ be an element of the Torelli group $\mathcal{I}_{g,1}$, namely φ acts trivially on the first homology group $H_1(\Sigma_{g,1}, \mathbb{Z})$. Then we notice that $\log \tau_1(M_\varphi) = 0$ holds for any $\varphi \in \mathcal{I}_{g,1}$ (see Corollary 5.3). To give an explicit formula of $\log \tau_2$, we consider the Magnus representation

$$r_2 : \mathcal{M}_{g,1} \rightarrow GL(2g, \mathbb{Z}N_2),$$

where $N_2 = \Gamma/[\Gamma, \Gamma] \cong H_1(\Sigma_{g,1}, \mathbb{Z})$. If we restrict r_2 to the Torelli group $\mathcal{I}_{g,1}$, this is really a homomorphism (see [23] Corollary 5.4). Then our formula for the second L^2 -torsion invariant is the following. The proof is similar to one for Theorem 5.1.

Theorem 5.10 ([12]). *For any mapping class $\varphi \in \mathcal{I}_{g,1}$, the logarithm of the second L^2 -torsion invariant $\tau_2(M_\varphi)$ is given by*

$$\log \tau_2(M_\varphi) = -2m(\Delta_{r_2(\varphi)}),$$

where $\Delta_{r_2(\varphi)}(y_1, \dots, y_{2g}, t) = \det A_2 = \det(tI - \overline{r_2(\varphi)})$ and y_i denotes the homology class corresponding to x_i .

Now we suppose $F(\mathbf{t}) \in \mathbb{Z}[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$ is primitive. We define F to be a generalized cyclotomic polynomial if it is a monomial times a product of one-variable cyclotomic polynomials evaluated at monomials.

The next corollary immediately follows from the theorem of Boyd, Lawton and Smyth (see [6] Theorem 4).

Corollary 5.11. *For any mapping class $\varphi \in \mathcal{I}_{g,1}$, $\log \tau_2(M_\varphi) = 0$ if and only if $\Delta_{r_2(\varphi)}$ is a generalized cyclotomic polynomial.*

As a typical element of the Torelli group $\mathcal{I}_{g,1}$, we first consider a BSCC-map φ_h ($1 \leq h \leq g$) of genus h . That is, a Dehn twist along a bounding simple closed curve on $\Sigma_{g,1}$ which separates $\Sigma_{g,1}$ into $\Sigma_{h,1}$ and genus $g-h$ surface with two boundaries. We then see from [25] Corollary 4.3 that $\Delta_{r_2(\varphi_h)} = (t-1)^{2g}$. This is clearly a generalized cyclotomic polynomial, so that $\log \tau_2(M_{\varphi_h}) = 0$.

Second we consider a BP-map $\psi_h = D_c D_{c'}^{-1}$ of genus h ($1 \leq h \leq g-1$), where c and c' are disjoint homologous simple closed curves on $\Sigma_{g,1}$ and D_c denotes the Dehn twist along c . Since

$$\Delta_{r_2(\psi_h)} = (t-1)^{2g-2h}(t-y_{g+h+1})^{2h}$$

holds (see [25]), where y_{g+h+1} denotes the homology class corresponding to the $(h+1)$ th meridian of $\Sigma_{g,1}$, we also have $\log \tau_2(M_{\psi_h}) = 0$.

The next example shows the non-triviality of the second L^2 -torsion invariant $\log \tau_2$.

Example 5.12. Let $\varphi = t_3 \varphi_1 t_3^{-1} \varphi_1 \in \mathcal{I}_{2,1}$. Then we see from a computation in [25] that

$$\Delta_{r_2(\varphi)} = (t-1)^4 + t(t-1)^2(y_1 - 2 + y_1^{-1})(y_2 - 2 + y_2^{-1}).$$

This is not a generalized cyclotomic polynomial, so that the mapping torus M_φ has a non-trivial L^2 -torsion invariant $\tau_2(M_\varphi)$. In fact we can numerically compute it by means of Lawton's result (see [13]). More precisely we have

$$\begin{aligned} -3\pi \log \tau_2(M_\varphi) &= 6\pi m(\Delta_{r_2(\varphi)}) \\ &= 6\pi \lim_{r \rightarrow \infty} m(\Delta_{r_2(\varphi)}(u, u, u^r)) \\ &= 19.28\dots \end{aligned}$$

6. VANISHING OF $\log \tau_k$ FOR REDUCIBLE MAPPING CLASSES

From the Nielsen-Thurston theory (see [5]), the mapping classes of a surface are classified into the following three types: (i) periodic, (ii) reducible and (iii) pseudo-Anosov. In our point of view, the most interesting object is a pseudo-Anosov map φ . Because the corresponding mapping torus M_φ has non-trivial hyperbolic volume.

In this final section, we show two vanishing theorems for $\log \tau_k$. We introduced an infinite sequence $\{\tau_k\}$ as an approximation of the hyperbolic volume. Thus if it behaves well with the index k , we ought to prove

$$\lim_{k \rightarrow \infty} \log \tau_k = 0$$

for non-hyperbolic 3-manifolds (see Problem 4.4). As a first step of this observation, we obtain the following.

Theorem 6.1. *If $\varphi \in \mathcal{M}_{g,1}$ is the product of Dehn twists along any disjoint non-separating simple closed curves on $\Sigma_{g,1}$ which are mutually non-homologous, then $\log \tau_k(M_\varphi) = 0$ for any $k \geq 1$.*

Remark 6.2. The mapping torus M_φ for $\varphi \in \mathcal{M}_{g,1}$ as above admits no hyperbolic structures, so that $\text{vol}(M_\varphi) = 0$ holds.

Proof. At first, we prove the theorem for the genus one case. After that we give the outline of the proof in the higher genus case.

Let D_c be a Dehn twist along a non-separating simple closed curve c on $\Sigma_{1,1}$. Taking a conjugation, we can assume that the curve c is one of the standard generators of $\pi_1(\Sigma_{1,1})$. We then see that $\varphi = D_c^q$ is represented by a matrix

$$\begin{pmatrix} 1 & q \\ 0 & 1 \end{pmatrix} \in SL(2, \mathbb{Z}).$$

Thus we can choose a deficiency one presentation

$$\langle x, y, t \mid txt^{-1} = x, tyt^{-1} = x^q y \rangle$$

of $\pi_1(M_\varphi)$. Applying the free differential calculus to the relators $txt^{-1}x^{-1}$ and $tyt^{-1}(x^q y)^{-1}$, we obtain the Alexander matrix

$$A = \begin{pmatrix} t-1 & 0 \\ -\partial(x^q)/\partial x & t-x^q \end{pmatrix}.$$

Here we remark that the generators t and x can be commuted by the relation $txt^{-1} = x$. Hence in this case, the k th Alexander matrix A_k coincides with the original matrix A . In particular, t and x always commute. As we saw in Section 5, the L^2 -torsion invariant $\tau_k(M_\varphi)$ ($k \geq 1$) can be computed by using the usual determinant and the Mahler measure in such a situation. Since

$$\det A = (t - 1)(t - x^q)$$

is a generalized cyclotomic polynomial, we obtain $\log \tau_k(M_\varphi) = 0$ as desired (see Corollary 5.11).

In the higher genus case, we can assume that the mapping class φ is given by

$$\begin{aligned} \varphi_*(x_1) &= x_1, \quad \varphi_*(x_2) = x_1^{q_1} x_2, \dots, \\ \varphi_*(x_{2l-1}) &= x_{2l-1}, \quad \varphi_*(x_{2l}) = x_{2l-1}^{q_l} x_{2l}, \\ \varphi_*(x_{2l+1}) &= x_{2l+1}, \dots, \varphi_*(x_{2g}) = x_{2g} \end{aligned}$$

by taking a conjugation, where $q_1, \dots, q_l \in \mathbb{Z}$ and $1 \leq l \leq g$. We then obtain the following presentation of $\pi_1(M_\varphi)$:

$$\langle x_1, \dots, x_{2g}, t \mid tx_i t^{-1} = \varphi_*(x_i), \ 1 \leq i \leq 2g \rangle.$$

Since the Alexander matrix A is the direct sum of the 2×2 -matrix in the genus one case, so that we obtain $\log \tau_k(M_\varphi) = 0$ by the similar arguments. \square

As another affirmative answer to Problem 4.4, we can show the vanishing of $\log \tau_k$ for the following mapping classes (see [12]). That is, we consider the case where there exists an integer n such that M_{φ^n} is topologically the product of $\Sigma_{g,1}$ and S^1 . Here its bundle structure is non-trivial in general. Namely the n th power φ^n of a given monodromy φ is not trivial. A typical example is the Dehn twist along the simple closed curve on $\Sigma_{g,1}$ parallel to the boundary. The difference between an isotopy fixing the boundary pointwisely and such one setwisely, it gives birth to the difference between a bundle structure and a topological type. We then obtain

Theorem 6.3 ([12]). $\log \tau_k(M_\varphi) = 0$ for any $k \geq 1$.

It is easy to see that such a 3-manifold does not admit a hyperbolic structure. Hence it has trivial simplicial volume.

The above two examples are both non-hyperbolic cases, so that we conclude the present paper with the following problem.

Problem 6.4. *Show*

$$\lim_{k \rightarrow \infty} \log \tau_k(M_\varphi) = \log \tau(M_\varphi)$$

for a pseudo-Anosov diffeomorphism φ .

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